On the primeness of 2-permutability

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Algebras and varieties

Definition

An **algebra** $\mathbf{A} = (A; \mathcal{F})$ is a non-empty base set A together with a set $\mathcal{F}$ of finitary operations on A.

The two-element Boolean algebra $\mathbf{B}_2 = (\{0, 1\}; \wedge, \vee, ', 0, 1)$ has two binary, one unary and two nullary operations. The alternating group $\mathbf{A}_5 = (A_5; \cdot, ^{-1}, \mathrm{id})$ has one binary, one unary and one nullary operation.

Definition

A **variety** $\mathcal{K}$ is a class of algebras with the same signature $\mathcal{F}$ that are precisely the models of a set Γ of defining equations over $\mathcal{F}$.

The variety $\mathcal{G}$ of groups is defined by the following equations

$$\Gamma = \{x \cdot (y \cdot z) \approx (x \cdot y) \cdot z, \mathrm{id} \cdot x \approx x \cdot \mathrm{id} \approx x, x \cdot x^{-1} \approx x^{-1} \cdot x \approx \mathrm{id}\}.$$

Boolean algebras and boolean rings

Example

The variety $\mathcal{BA}$ is the class of all Boolean algebras $(B; \wedge, \vee, ', 0, 1)$ defined by the usual associativity, commutativity, distributivity, absorption, identity and complements equations.

Example

The variety $\mathcal{BR}$ is the class of all Boolean rings $(R; +, \cdot, 0, 1)$ defined by the usual associativity, commutativity, distributivity, identity and idempotency equations.

The two varieties are not the same, as they have different operational symbols and thus different algebras, but they are equivalent. Every Boolean algebra can be turned into a boolean ring and vice versa:

$$x + y = (x \wedge \neg y) \vee (\neg x \wedge y), \quad x \cdot y = x \wedge y, \quad 1 = 1, \quad 0 = 0.$$

Interpretation of varieties

Definition (W.D. Neumann; 1974)

Let Γ be a set of identities over a signature. We say that Γ **interprets in a variety** $\mathcal{K}$ if by replacing the operation symbols in Γ by some term expressions of $\mathcal{K}$, the so obtained set of identities holds in $\mathcal{K}$.

Definition

A **variety** $\mathcal{K}_1$ **interprets in a variety** $\mathcal{K}_2$, denoted as $\mathcal{K}_1 \preceq \mathcal{K}_2$, if there is a set of identities Γ that defines $\mathcal{K}_1$ and interprets in $\mathcal{K}_2$.

- The variety $\mathcal{G}$ of groups interprets in the variety $\mathcal{A}G$ of abelian groups.
- The varieties of Boolean algebras and rings are equi-interpretable.
- The variety of sets interprets in any other variety.
- Every variety interprets in the variety of trivial algebras ($x \approx y$).
- Constants c are modelled by unary operations satisfying $c(x) \approx c(y)$.
- The iterpretability relation $\preceq$ is a quasi-order on the class of varieties.

Lattice of interpretability types

Theorem

The class of varieties modulo equi-interpretability forms a bounded lattice, **the lattice of interpretability types**, *with* $\overline{\mathcal{V}} \vee \overline{\mathcal{W}} = \overline{\mathcal{V} \amalg \mathcal{W}}$ *and* $\overline{\mathcal{V}} \wedge \overline{\mathcal{W}} = \overline{\mathcal{V} \otimes \mathcal{W}}$.

Definition

The **coproduct** of the varieties $\mathcal{V} = \operatorname{Mod}\Sigma$ and $\mathcal{W} = \operatorname{Mod}\Delta$ in disjoint signatures is the variety $\mathcal{V} \amalg \mathcal{W} = \operatorname{Mod}(\Sigma \cup \Delta)$.

Definition

The **varietal product** of $\mathcal{V}$ and $\mathcal{W}$ is the variety $\mathcal{V} \otimes \mathcal{W}$ of algebras $\mathbf{A} \otimes \mathbf{B}$ for $\mathbf{A} \in \mathcal{V}$ and $\mathbf{B} \in \mathcal{W}$ whose

- universe is $A \times B$,
- basic operations are $s \otimes t$ acting coordinate-wise for each pair of n-ary terms of $\mathcal{V}$ and $\mathcal{W}$.

O. Garcia, W. Taylor (1984): Lattice of interpretability types of varieties

- minimal element: sets (equi-interpretable with semigroups)
- maximal element: trivial algebras
- the class of idempotent varieties form a sublattice
- the class of finitely presented varieties forms a sublattice
- the class of varieties defined by linear equations forms a join sub-semilattice
- not modular
- meet prime elements: boolean algebras, lattices, semilattices
- meet irreducible elements: groups
- join prime elements: commutative groupoids, trivial algebras

J. Mycielski (1977): Lattice of interpretability types of first order theories

- local interpretability
- distributive

Congruence permutability

Definition

A variety $\mathcal{V}$ is **congruence** n**-permutable** ($n \geq 2$) if every algebra $\mathbf{A} \in \mathcal{V}$ satisfies $\alpha \circ^n \beta = \beta \circ^n \alpha$ for all congruences $\alpha, \beta \in \operatorname{Con} \mathbf{A}$.

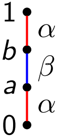
Theorem (A.I. Maltsev; 1954)

A variety $\mathcal{V}$ is congruence 2-permutable, that is $\alpha \circ \beta = \beta \circ \alpha$, if and only if it has a ternary term m satisfying $m(x,x,y) \approx m(y,x,x) \approx y$.

- Examples: groups $m(x,y,z) = xy^{-1}z$, rings, vector spaces.

$$(x,z) \in \alpha \circ \beta \Rightarrow x \; \alpha \; y \; \beta \; z \Rightarrow x \; \beta \; m(x,y,z) \; \alpha \; z \Rightarrow (x,z) \in \beta \circ \alpha$$

- The variety of lattices is not congruence n-permutable for any n:



$$\begin{aligned} \alpha &= \{(0,a),(b,1),\dots\}, \\ \beta &= \{(a,b),\dots\}, \\ (0,1) &\in \alpha \circ \beta \circ \alpha, \\ (0,1) &\notin \beta \circ \alpha \circ \beta. \end{aligned}$$

Theorem (J. Hagemann, A. Mitschke; 1973)

For a variety $\mathcal{V}$ and $n \geq 2$ the following are equivalent:

- *$\mathcal{V}$ is congruence n-permutable,*
- *$\varrho^{-1} \subseteq \varrho \circ^{n-1} \varrho$ for any $\mathbf{A} \in \mathcal{V}$ and reflexive relation $\varrho \leq \mathbf{A}^2$,*
- *$\mathcal{V}$ has ternary terms $p_1, \ldots, p_{n-1}$ satisfying*

$$
\begin{aligned}
x &\approx p_1(x, y, y),\\
p_i(x, x, y) &\approx p_{i+1}(x, y, y) \text{ for } 1 \leq i < n-1,\\
p_{n-1}(x, x, y) &\approx y.
\end{aligned}
$$

Definition

A **compatible digraph of an algebra A** is a digraph whose vertex set coincides with the base set of **A** and whose edge relation is preserved by all of the basic operations of **A**. A **compatible digraph in a variety** is a compatible digraph of an algebra in the variety.

- 2-permutability: every compatible reflexive digraph is symmetric.

Primeness and a conjecture

- The types of the varieties in which a given set of equations interprets constitute a principal filter in the lattice of interpretability types.
- A **filter is prime** if none of the joins of any two elements from its complement belongs to it.
- The importance of prime principal filters in the lattice of interpretability types.

Conjecture (O. Garcia, W. Taylor; 1984)

2-permutability is prime in the lattice of interpretability types of varieties.

Question

For a fixed $n \geq 2$ does n-permutability determine a prime filter in the lattice of interpretability types?

Related results

- In 1996 S. Tschantz announced a proof of the conjecture. However, his proof has remained unpublished.
- K. Kearnes and S. Tschantz: 2-permutability is prime in the lattice of interpretability types of **idempotent** varieties. There are no similar results known for n-permutability when $n > 2$.
- In his PhD thesis, Sequeira proved the result that 2-permutability is prime in the lattice of interpretability types of **linear** varieties.
- Opršal: for any $n \geq 2$, n-permutability is prime in the lattice of interpretability types of **linear** varieties.

Theorem (G. Gyenizse, M. Maróti and L. Zádori; 2020)

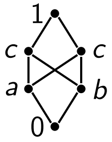
For any $n \geq 5$, n-permutability is **not prime** *in the lattice of interpretability types of varieties.*

Theorem (G. Gyenizse, M. Maróti and L. Zádori; 2022)

2-permutability **is prime** *in the lattice of interpretability types of varieties.*

The proof of the n-permutable result when $n \geq 5$

- We took the variety $\mathcal{M}$ defined by the majority identities $m(y,x,x) \approx m(x,y,x) \approx m(x,x,y) \approx x$ and a variety $\mathcal{O}$ generated by the following order primal algebra and we proved that $\mathcal{M} \vee \mathcal{O}$ is 5-permutable.



- It is not known if there exists a non-3-permutable variety $\mathcal{O}$ for which $\mathcal{M} \vee \mathcal{O}$ is 3-permutable. We proved that such a variety cannot be locally finite.
- It turned out that for any non-2-permutable variety $\mathcal{O}$, $\mathcal{M} \vee \mathcal{O}$ is not 2-permutable.

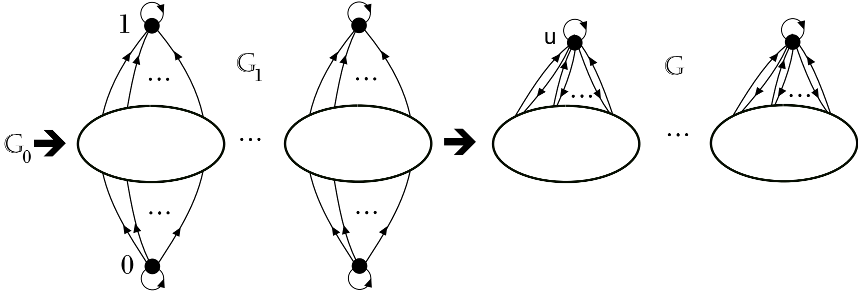
$\mathcal{M} \vee \mathcal{O}$ for a non-2-permutable variety $\mathcal{O}$

Lemma 1.

The join of $\mathcal{M}$ and a non-2-permutable variety $\mathcal{O}$ is non-2-permutable.

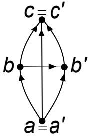
- It suffices to construct an algebra $\mathbf{A}$ that has basic operations satisfying the identities of $\mathcal{O}$ and a majority basic operation such that $\mathbf{A}$ has no Maltsev term operation.
- It suffices to construct a compatible digraph $\mathbb{G}$ in $\mathcal{O}$ such that $\mathbb{G}$ admits a majority operation and admits no Maltsev operation.
- By the Hagemann-Mitschke theorem, there is a non-symmetric compatible reflexive digraph $\mathbb{G}_0$ in $\mathcal{O}$. We define a compatible digraph $\mathbb{G}_1$ from $\mathbb{G}_0$ and then the compatible digraph $\mathbb{G}$ from $\mathbb{G}_1$ in $\mathcal{O}$.

Proof of the Lemma 1: properties of $\mathbb{G}_1$ and $\mathbb{G}$



Proof of the Lemma 1: Part 1

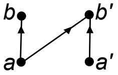
- Let $\mathbb{G}_1$ be the digraph whose vertex set consists of the 3-tuples (a, b, c) where $a \to b \to c$ and $a \to c$ in $\mathbb{G}_0$, and whose edge relation is given by $(a, b, c) \to (a', b', c')$ iff $a = a'$, $c = c'$ and $b \to b'$ in $\mathbb{G}_0$.



- Every component C of $\mathbb{G}_1$ is of the form $\{\ (a, x, c) :\ a \to x \to c\ \}$ for some $a \to c$. Indeed, $(a, a, c) \in G_1$ as $\mathbb{G}_0$ is reflexive. Moreover, $(a, a, c) \to (a, x, c)$ for all x with $a \to x \to c$. Similarly, $(a, x, c) \to (a, c, c)$.
- Thus, in every component C of $\mathbb{G}_1$ there are special elements 0_C and 1_C such that for all $z \in C$, $0_C \to z \to 1_C$.
- Moreover, since $\mathbb{G}_0$ is not symmetric, there is some $a \to c$ and $c \not\to a$ in $\mathbb{G}_0$. Hence $(a, c, c) \not\to (a, a, c)$ in $\mathbb{G}_1$. So there is a component C with $1_C \not\to 0_C$ in $\mathbb{G}_1$.

Proof of the Lemma 1: Part 2

- Let $\mathbb{G}$ be the digraph whose vertex set consists of the pairs (a, b) where $a \to b$ in $\mathbb{G}_1$, and whose edge relation is given by $(a, b) \to (a', b')$ iff $a \to b'$ in $\mathbb{G}_1$.



- Let C be a component of $\mathbb{G}$. Notice that there is a component D of $\mathbb{G}_1$ such that for all $(a, b) \in C$, $a, b \in D$. Let 0 and 1 be the special elements from D. Clearly, $(0, 1) \to (a, b) \to (0, 1)$ for all $(a, b) \in C$. Thus $(0, 1)$ is a universal vertex of C.

- There is some non-complete component of $\mathbb{G}$. Indeed, let $1 \not\to 0$ in some component D of $\mathbb{G}_1$, then we have $(0, 0) \to (1, 1) \not\to (0, 0)$ in $\mathbb{G}$.

Proof of the Lemma 1: Part 3

- For every component C with universal vertex u_C in $\mathbb{G}$, we define the partial operation m_C on C^3 by
  $m_C(y,x,x) = m_C(x,y,x) = m_C(x,x,y) = x$.
  Clearly, m_C extends to C^3 as a majority operation by using the constant value u_C. The partial majority operation from the union of the C^3 extends to a full majority operation on G^3 via a suitable projection on every remaining component of G^3.
- Assume that f is a Maltsev operation admitted by $\mathbb{G}$, C is a component of $\mathbb{G}$, u is a universal vertex in C and a and b are two vertices in C such that $a \not\to b$.
- Then we have that $a \to u \to u \to b$ in C, and by applying f to these edges, we get $a = f(a,u,u) \to f(u,u,b) = b$ in C, a contradiction.

Corollary

A variety is 2-permutable iff every compatible digraph, each of whose components has a universal vertex, is a disjoint union of complete digraphs.

The Reciprocity Lemma for digraphs with a universal vertex

- The **complement** of a digraph $\mathbb{G}$ is the digraph whose vertex set is G and whose edge set is $G^2 \setminus E(\mathbb{G})$.
- The **product** of the digraphs $\mathbb{G}$ and $\mathbb{H}$ is defined on $G \times H$ with $(g_1, h_1) \to (g_2, h_2) \iff g_1 \to g_2$ and $h_1 \to h_2$.
- We define the **power** $\mathbb{G}^{\mathbb{H}}$ of two digraphs as follows. The vertex set of $\mathbb{G}^{\mathbb{H}}$ is the set of maps from the set H to the set G. The edge relation of $\mathbb{G}^{\mathbb{H}}$ is defined by

$$f \to f' \text{ if and only if } f(x) \to f'(y) \text{ in } \mathbb{G} \text{ for every } x \to y \text{ in } \mathbb{H}.$$

Lemma 2. (Reciprocity Lemma)

Let $\mathbb{G}$ and $\mathbb{H}$ be two digraphs with universal vertices u_G and u_H, respectively. Let $\mathbb{G}^*$ and $\mathbb{H}^*$ be the complements of the digraphs $\mathbb{G} - u_G$ and $\mathbb{H} - u_H$. Let κ be an infinite cardinal where $\kappa \geq \max(|G|, |H|)$, and let $\mathbb{K}$ be a complete digraph of κ-many vertices. Then

$$\mathbb{G}^{\mathbb{H}^* \times \mathbb{K}} \cong \mathbb{H}^{\mathbb{G}^* \times \mathbb{K}}.$$

Proof the the main theorem: Part 1

- **Main Theorem: 2-permutability is a prime Maltsev condition.**
- It suffices to prove that the join of any two non-permutable varieties $\mathcal{K}$ and $\mathcal{L}$ is non-permutable.
- It suffices to construct a digraph that is compatible in both varieties and admits no Maltsev operation.
- By the Corollary, there are compatible digraphs $\mathbb{G}$ in $\mathcal{K}$ and $\mathbb{H}$ in $\mathcal{L}$ such that both of $\mathbb{G}$ and $\mathbb{H}$ have a non-complete component and each component of $\mathbb{G}$ and $\mathbb{H}$ has a universal vertex.
- First, assume that $\mathbb{G}$ and $\mathbb{H}$ are connected. So, both digraphs have a universal vertex and are non-complete.

Proof the the main theorem: Part 2

- By the Reciprocity Theorem, $\mathbb{G}^{\mathbb{H}^* \times \mathbb{K}} \cong \mathbb{H}^{\mathbb{G}^* \times \mathbb{K}}$. So $\mathbb{G}^{\mathbb{H}^* \times \mathbb{K}}$ is a compatible digraph in both $\mathcal{K}$ and $\mathcal{L}$.
- As there are $h, h' \in H^*$ such that $h \not\to h'$ in $\mathbb{H}$, the constant maps span a subdigraph isomorphic to $\mathbb{G}$ in $\mathbb{G}^{\mathbb{H}^* \times \mathbb{K}}$.
- Thus, $\mathbb{G}^{\mathbb{H}^* \times \mathbb{K}}$ is a non-complete digraph with a universal vertex.
- By the Corollary, $\mathbb{G}^{\mathbb{H}^* \times \mathbb{K}}$ admits no Maltsev operation. So the proof is done, if $\mathbb{G}$ and $\mathbb{H}$ are connected.

Proof the the main theorem: Part 3 (a sketch)

- We construct new digraphs from $\mathbb{G}$ in $\mathcal{K}$ with the properties below.
- $\mathbb{G}$: each component has a universal vertex + at least one component is non-complete
- $\mathbb{G}_1$: + at least one component is complete
- $\mathbb{G}_2$: + the cardinality of each component equals the number of complete components and the number of non-complete components.
- $\mathbb{G}_3$: + the non-complete components all are isomorphic to some digraph $\mathbb{T}$.
- One can construct $\mathbb{H}_1$, $\mathbb{H}_2$ and $\mathbb{H}_3$ from $\mathbb{H}$ in $\mathcal{L}$ with the same $\mathbb{T}$.
- Then $\mathbb{G}_3 \cong \mathbb{H}_3$ is compatible in $\mathcal{K}$ and $\mathcal{L}$ and admits no Maltsev operation. Q.E.D.